PHYSICAL REVIEW E VOLUME 59, NUMBER 4 APRIL 1999

Unstable dimension variability and complexity in chaotic systems

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(Received 22 October 1998)

We examine the interplay between complexity and unstable periodic orbits in high-dimensional chaotic systems. Argument and numerical evidence are presented suggesting that complexity can arise when the system is severely nonhyperbolic in the sense that periodic orbits with a distinct number of unstable directions coexist and are densely mixed. A quantitative measure is introduced to characterize this unstable dimension variability. [S1063-651X(99)51404-9]

PACS number(s): 05.45.Ac

Complexity has become an issue of recent interest [1]. While there is no general definition of complexity at present, previous works suggest that complex systems should have the following three properties: (1) they consist of many components that are interconnected in a complicated manner, (2) the components can be either regular or irregular, and (3) the components exist on different length and/or time scales, i.e., a complex system exhibits a hierarchy of structures [1–6]. These are also called the *three traits* of a complex system [1–5]. Complex systems are common in many natural systems such as the Rayleigh-Bénard convection [7], neuronal activity [8], extended nonlinear optical systems [9], fluidized beds [10], etc. One manifestation of complexity in chaotic systems is the basin structure. Take, for example, the double rotor map: a mechanical system of two degrees of freedom subject to external periodic kick [11]. The map exhibits all three traits of a complex system in wide parameter regimes [5]. Often, there are many coexisting periodic attractors whose basins of attraction are interconnected via chaotic saddles on the boundaries in a complex way, and the basin boundaries permeate most of the phase space.

In order to better understand and manipulate complex systems, it is important to study the dynamical origin of complexity. In particular, one wishes to understand how complexity arises in terms of the fundamental dynamical quantities of the underlying system. And there is nothing more fundamental than studying complexity in terms of the *unstable periodic orbits* embedded in the invariant set, which are commonly regarded as the basic building blocks of a nonlinear systems [12]. The aim of this paper is to present explicit evidence elucidating the interplay between complexity and unstable periodic orbits in high-dimensional chaotic systems. Our principal result is that complexity can be accompanied by a particular type of nonhyperbolicity: *unstable dimension variability* [13–16]. Roughly, unstable dimension variability means that unstable periodic orbits embedded in a chaotic set have distinct numbers of unstable directions. As a consequence, a trajectory typically moves in regions with different unstable dimensions, leading to fluctuations about zero of some Lyapunov exponents [14,15]. We introduce a quantity, a *contrast* measure, to characterize the degree of unstable dimension variability. We then study a system, for which a subclass of unstable periodic orbits can be computed explicitly, to demonstrate the existence of extremely complicated basin boundaries among multiple coexisting attractors in parameter regimes where there is severe unstable dimension variability. Our study represents one approach to study complexity in terms of dynamical quantities that are well understood in the context of low-dimensional chaotic systems [17], knowledge of which has now become relatively ample.

Before we describe the detail of our work, we wish to stress that unstable dimension variability provides only *one* possible mechanism for complexity. The description of complexity in terms of the three traits, as stated above, is mathematically inexact, while unstable dimension variability is a nonhyperbolic property of chaotic systems that can be defined rigorously [13–16]. There are undoubtedly many other scenarios to complexity that need to be explored. The main motivation for us to study unstable dimension variability is that it has been considered as a fundamental dynamical property of high-dimensional chaotic systems [13–16] where complicated behaviors, such as riddled basins [18], can arise. Even then, as we will see in numerical examples, unstable dimension variability alone cannot guarantee complexity. For instance, another necessary condition for riddled basins to occur is that there must be multiple coexisting attractors in the phase space. Complex systems can also exhibit highly simplified global behavior in certain parameter ranges [1–6]. At present, it is not clear whether unstable dimension variability can account for such simple behaviors.

To argue that unstable dimension variability can lead to physically observable, complex behaviors, we consider the system of N coupled d-dimensional maps. Assume that the system has a synchronization manifold $\mathcal{M}$ [19,20], and the individual maps, when decoupled, exhibit a chaotic attractor with one positive Lyapunov exponent that is not close to zero so that there is no unstable dimension variability. Note that this chaotic attractor is the one in $\mathcal{M}$. Since such a system of coupled maps represents a spatiotemporal system, a systematic analysis of unstable periodic orbits in the full phase space is extremely difficult. We thus focus on unstable periodic orbits in $\mathcal{M}$. The local eigenspace of each periodic orbit in $\mathcal{M}$ is Nd dimensional, and each orbit has at least one unstable direction, the one associated with the chaotic attractor in $\mathcal{M}$. There are $N-1$ subspaces that are transverse to $\mathcal{M}$: each is d dimensional. For $\epsilon=0$ (uncoupled case), each transverse subspace is unstable and, hence, all periodic orbits have N unstable directions and there is no unstable dimension variability. In this case, $\mathcal{M}$ is transversely unstable and the asymptotic attractor of the system is a trivial combination of N chaotic attractors of the individual maps. There is thus

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no difficulty of predicting it for almost all initial conditions chosen from its basin. As the coupling ϵ is increased from zero, a periodic orbit in $\mathcal{M}$ can become stable in some of its transverse subspaces, leading to unstable dimension variability. To see how complexity can arise, imagine the situation where there are slightly more periodic orbits that are stable in all transverse subspaces than those that are unstable in at least one transverse subspace, so that, on average, a chaotic trajectory in $\mathcal{M}$ is transversely stable. Call the former set Σ_{TS} and the latter set Σ_{TU}. Initial conditions in the vicinity of a periodic orbit in Σ_{TS}, but off $\mathcal{M}$, can be attracted towards $\mathcal{M}$ and will asymptote towards it. For these initial conditions, the asymptotic attractor is the chaotic one in $\mathcal{M}$ and synchronization can be achieved for these initial conditions in a noiseless situation [20]. An initial condition near one of the periodic orbits in Σ_{TU}, however, can be repelled away temporally from $\mathcal{M}$ along one of the transversely unstable directions. If there are other attractors coexisting with the one in $\mathcal{M}$ in the phase space, there is a nonzero probability that such an initial condition can asymptote towards one of the attractors that are not in $\mathcal{M}$. Since Σ_{TS} and Σ_{TU} are typically intermingled in the chaotic attractor in $\mathcal{M}$, practically, it is not possible to trace a specific initial condition to its asymptotic attractor. In this case, the basin of the attractor in $\mathcal{M}$ is riddled [18]. Regarding each attractor, together with its basin, as a component of the system of coupled maps, we see that complexity can arise as a consequence of unstable dimension variability.

In order to explicitly demonstrate the interplay between unstable dimension variability and complexity, we study the following system of globally coupled Hénon maps:

$$x_{n+1}(i)=a-\left[(1-\epsilon)x_n(i)+\frac{\epsilon}{N-1}\sum_{j,j\neq i}^{N}x_n(j)\right]^2+by_n(i),$$
$$y_{n+1}(i)=x_n(i),\quad i=1,\ldots,N, \tag{1}$$

where the synchronization state $\mathcal{M}$, $x(1)=\cdots=x(N)$, is a solution of Eq. (1). In $\mathcal{M}$, Eq. (1) reduces to the two-dimensional Hénon map [21] for which a and b are parameters. Unstable periodic orbits of the Hénon map can be explicitly computed with high precision by using the method in Ref. [22]. In our numerical experiments, we choose $a=1.4$ and $b=0.3$, the standard parameter setting for which the Hénon map exhibits a chaotic attractor for most initial conditions chosen from the region $-2\leq(x,y)\leq 2$. There is at least another attractor, the attractor at $-\infty$. The boundary between the basins of the chaotic attractor and the one at infinity is apparently smooth. Thus, for the single Hénon map, there is no complexity, which characterizes the dynamics in $\mathcal{M}$. For $\epsilon=0$, Eq. (1) reduces to a set of N isolated Hénon maps, and there is no complexity. In this case, every periodic orbit embedded in the chaotic attractor in $\mathcal{M}$ is unstable in all $(N-1)$ transverse subspaces and, hence, there is no unstable dimension variability either. To characterize the global dynamical behavior of Eq. (1) as ϵ is increased from zero, we compute the Lyapunov spectrum of Eq. (1). Figure 1 shows, for $N=2$, the four Lyapunov exponents versus ϵ, where 1000 values of ϵ are uniformly distributed in the interval [0.15, 0.25] and for each ϵ, 10^7 iterations (with 10^6 preiterations) are used to compute the exponents. We note that there are wild fluctuations of the Lyapunov exponents.

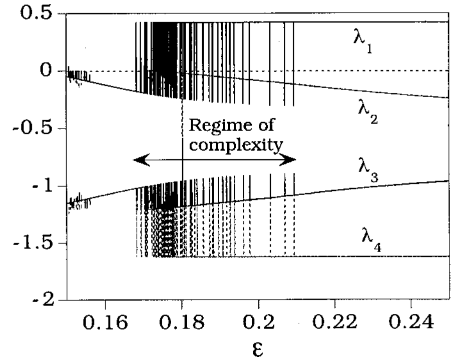

FIG. 1. The Lyapunov spectrum vs ϵ for $N=2$ in Eq. (1).

The fluctuations indicate the coexistence of multiple attractors, and the wildness of the fluctuations suggests that the boundaries between the basins of these attractors may be quite complicated [23]. Similar behavior persists for Eq. (1) for $N>2$ [23].

To see that complex behavior actually occurs in parameter regions where the Lyapunov exponents fluctuate widely, we examine the basin structure of the coexisting attractors. Figure 2 shows, for $N=2$ and $\epsilon=0.18$, the behavior of initial conditions chosen from the two-dimensional area $-1.0\leq[x(1),x(2)]\leq 1.0$ with $y(1)=y(2)=0$. The blue diagonal line denotes the synchronization manifold $\mathcal{M}$, and the green, yellow, and red regions are the basins of the attractors in $\mathcal{M}$, off $\mathcal{M}$, and at ∞, respectively. Regarding each attractor together with its basin as one component of the system, we see that (1) there are multiple components that are interconnected in a very complicated manner, (2) both regular (e.g., the attractor at ∞) and irregular (e.g., the chaotic attractor in $\mathcal{M}$) components coexist, and (3) there is a hierarchy of structure for these components [24]. These are the three traits characterizing complexity. Thus, for Eq. (1), complexity can occur.

We now examine the behavior of unstable periodic orbits in parameter regimes where there is complexity. In order to facilitate the counting of possibilities of different unstable directions, we choose $N=2$ in Eq. (1). We compute all periodic orbits embedded in the Hénon attractor in $\mathcal{M}$ up to period 30 and compute the Lyapunov spectrum for each orbit. Since $N=2$, a periodic orbit can have either one or two unstable directions: Those with one (two) unstable direction are the orbits that are transversely stable (unstable). Our principal finding is that unstable dimension variability appears to be severe for $0.16<\epsilon<0.22$—the parameter regime where there is complexity.

To quantify the severeness of unstable dimension variability, we introduce the following *contrast* [25] measure:

$$C_p=\left|\frac{\mu_2(p)-\mu_1(p)}{\mu_2(p)+\mu_1(p)}\right|, \tag{2}$$

where $\mu_{1,2}(p)\equiv\sum_{j=1}^{N_{1,2}(p)}e^{-\lambda_1^j(p)p}$, and $N_1(p)$ and $N_2(p)$ are the numbers of periodic orbits of period p with one and two unstable directions, respectively, $\lambda_1^j(p)$ is the largest Lyapunov exponent of the jth periodic orbit of period p, and the factor $e^{-\lambda_1^j(p)p}$ approximates the natural measure associated with this orbit [26]. The quantities $\mu_{1,2}(p)$ are then the

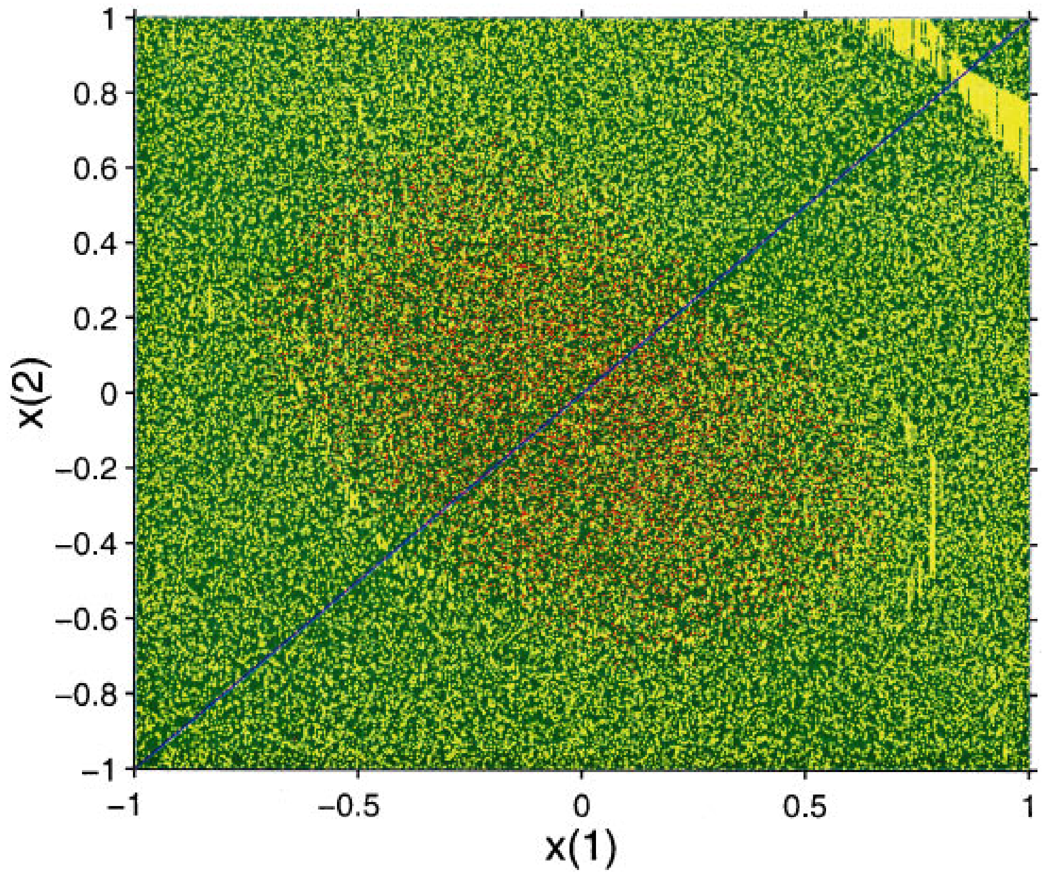

FIG. 2. (Color) Basins of attraction for attractors in $\mathcal{M}$ (green), off $\mathcal{M}$ (yellow), and at infinity (red) for $N=2$ and $\epsilon=0.18$ in Eq. (1).

weighted numbers of period-p orbits with one and two unstable directions, respectively. When there is no unstable dimension variability, we have either $N_2(p)=0$ or $N_1(p)=0$, which yields $C_p=1$. The contrast C_p starts to decrease from one when unstable dimension variability occurs, and the worst case is $C_p=0$, corresponding to the situation where unstable dimension variability is most severe $[\mu_1(p)=\mu_2(p)]$. Figure 3 shows C_{17} (filled circles), C_{20} (open circles), C_{25} (open squares), and C_{28} (open diamonds) versus ϵ. Apparently, for all four periods, C_p is minimum near

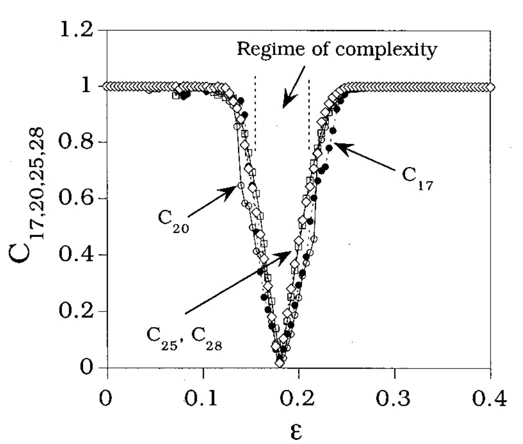

FIG. 3. The contrast vs ϵ for all periodic orbits of four different periods for $N=2$ in Eq. (1). The regime where the contrasts are well below 1 is the one in which severe unstable dimension variability occurs.

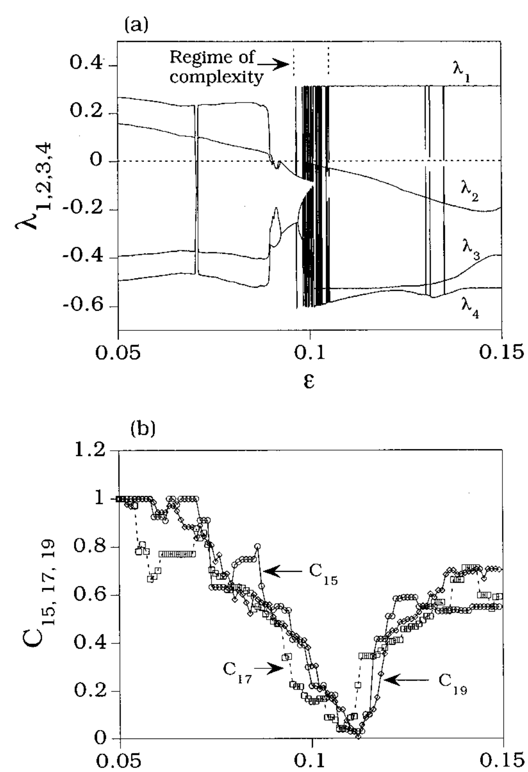

FIG. 4. (a) The Lyapunov spectrum vs the coupling strength ϵ and (b) the contrast measure C_p vs ϵ for $p=15$, 17, and 19 for the system of coupled Ikeda-Hammel-Jones-Moloney maps.

$\epsilon=0.18$, for which severe complexity is observed (Figs. 1 and 2). An examination of other periodic orbits of periods, say, larger than 15, reveals a similar behavior. Thus, the contrast measure C_p is not sensitive to the choice of the period p, insofar as p is large enough so that there are a large number of orbits. This can be understood by noting that C_p is a statistical quantity and, hence, it is meaningful only when a large number of unstable periodic orbits is involved. For small values of p, the number of orbits is usually too small for any reliable statistical estimate to be performed.

Can the correlation between the contrast measure C_p and complex behavior also be observed in any other systems? To address this question, we consider the following system of two coupled Ikeda-Hammel-Jones-Moloney maps [27]: $z_{n+1}=a+bz_ne^{i\phi_n}$, $z'_{n+1}=a+bz'_ne^{i\phi'_n}$, where $z=(x,y)$ and the coupling occurs in the phase variables ϕ_n and ϕ'_n: $\phi_n=k-p/(1+z_n^2)+2\pi\epsilon(x'_n-x_n)$, and $\phi'_n=k-p/[1+(z'_n)^2]+2\pi\epsilon(x_n-x'_n)$. Here, a, b, k, and p are parameters, and ϵ represents the coupling strength. The Ikeda-Hammel-Jones-Moloney map models the dynamics of an optical pulse propagating in a ring cavity, subject to partial reflection, phase and amplitude modulation, and distortion due to a nonlinear optical medium in the cavity. To search for complex behaviors, we choose $(a,b,k,p)=(0.85,0.9,0.4,5.18)$ so that the Ikeda-Hammel-Jones-Moloney map, in the absence of coupling, exhibits a chaotic attractor. As the coupling strength is increased from zero, we find a complicated basin structure near $\epsilon=0.1$, as suggested by Fig. 4(a), where the Lyapunov spectrum is plotted as a function of ϵ. To examine unstable dimension variability, we use the recent method reported in Ref. [28] to compute unstable periodic orbits embedded in the chaotic attractor in the synchronization manifold. Figure 4(b) shows the contrast measure C_p versus ϵ for $p=15$, 17, and 19. There is apparently a close correlation between complexity and unstable dimension variability. In particular, we see that complex behavior occurs when there is severe unstable dimension variability ($C_p<0.5$). Note that complex behavior seems to have disappeared when C_p reaches minimum at $\epsilon\approx0.11$. This is due to the disappearance of multiple coexisting attractors at $\epsilon\lesssim0.11$. This example also indicates that unstable dimension variability is, at most, a necessary condition for complexity.

In conclusion, we have examined the interplay between complexity and unstable periodic orbits by utilizing two systems of coupled chaotic maps. Explicit computation of periodic orbits and their Lyapunov spectra suggest that, in parameter regimes where there is complexity, the system can be extremely nonhyperbolic, as characterized by severe unstable dimension variability. This type of nonhyperbolicity may thus provide us with a hint to understand complex behaviors in deterministic chaotic systems.

I thank Dr. Y. Nagai and Dr. R. Davidchack for assiting in the computation of periodic orbits. This work was supported by the NSF under Grant No. PHY-9722156, and by the AFOSR under Grant No. F49620-98-1-0400.

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